IX. *On the Comparison of Hyperbolic Arcs.* *By* CHARLES W. MERRIFIELD. *Communicated by the Rev. Dr.* BOOTH, *F.R.S.*

Received March 3,—Read March 31, 1859.

AN application of JACOBI'S second theorem—the imaginary transformation—has led me to a formula which reduces the comparison of the arcs of hyperbolas to the same facility as that of elliptic arcs. The transformation is so easy and obvious, that I have had some hesitation in publishing it; but I observe that my result was not noticed by LEGENDRE, or by Professor MOSELEY, or in any more recent work which I have seen. Some of its applications, too, are worthy of remark.

I shall use the ordinary notations:—

$$\Delta(\theta, \varphi)=(1-\sin^2\theta\sin^2\varphi)^{\frac{1}{2}}, \quad F(\theta, \varphi)=\int\frac{d\varphi}{\Delta(\theta, \varphi)}, \quad E(\theta, \varphi)=\int\Delta(\theta, \varphi)d\varphi.$$

The functional equation

$$F\varphi_1+F\varphi_2-F\varphi_3=0 \quad \ldots \quad (A.)$$

is satisfied, as is well known, by either of the three trigonometrical equations—

$$\cos\varphi_3=\cos\varphi_1\cos\varphi_2-\sin\varphi_1\sin\varphi_2\sqrt{(1-\sin^2\theta.\sin^2\varphi_3)} \quad \ldots \quad (1.)$$

$$\cos\varphi_2=\cos\varphi_1\cos\varphi_3+\sin\varphi_1\sin\varphi_3\sqrt{(1-\sin^2\theta.\sin^2\varphi_2)} \quad \ldots \quad (2.)$$

$$\cos\varphi_1=\cos\varphi_2\cos\varphi_3+\sin\varphi_2\sin\varphi_3\sqrt{(1-\sin^2\theta.\sin^2\varphi_1)}. \quad \ldots \quad (3.)$$

Dividing each of these by $\cos\varphi_1.\cos\varphi_2.\cos\varphi_3$, and transposing, they become

$$\sec\varphi_3=\sec\varphi_1\sec\varphi_2+\tan\varphi_1\tan\varphi_2\sqrt{(1+\cos^2\theta.\tan^2\varphi_3)} \quad \ldots \quad (4.)$$

$$\sec\varphi_2=\sec\varphi_1\sec\varphi_3-\tan\varphi_1\tan\varphi_3\sqrt{(1+\cos^2\theta.\tan^2\varphi_2)} \quad \ldots \quad (5.)$$

$$\sec\varphi_1=\sec\varphi_2\sec\varphi_3-\tan\varphi_2\tan\varphi_3\sqrt{(1+\cos^2\theta.\tan^2\varphi_1)}. \quad \ldots \quad (6.)$$

It will be noticed, that we might pass from one set to the other, directly, by substituting $\sec\varphi$ for $\cos\varphi$, $\sqrt{-1}\tan\varphi$ for $\sin\varphi$, and $\cos\theta$ for $\sin\theta$. These substitutions constitute JACOBI'S second theorem. They convert

$$\frac{d\varphi}{(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}} \text{ into } \frac{\sqrt{-1}.d\varphi}{(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}},$$

and

$$(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}d\varphi \text{ into } \frac{\sqrt{-1}.d\varphi}{\cos^2\varphi}(1-\sin^2\theta.\sin^2\varphi)^{\frac{1}{2}}.$$

Now, calling $\int(1-\sin^2\theta_1\sin^2\varphi)^{\frac{1}{2}}d\varphi$, $E\varphi$, we know that

$$E\varphi_1+E\varphi_2-E\varphi_3=\sin^2\theta\sin\varphi_1.\sin\varphi_2.\sin\varphi_3. \quad \ldots \quad (7.)$$

If, therefore, we make the above substitutions in this equation, and divide by $\sqrt{-1}$, we have, making $H\phi=\int\frac{d\phi}{\cos^2\phi}(1-\sin^2\theta.\sin^2\phi)^{\frac{1}{2}}$,

$$H\phi_1+H\phi_2-H\phi_3=-\cos^2\theta.\tan\phi_1.\tan\phi_2.\tan\phi_3. \quad (8.)$$

Moreover, since we also have

$$F\phi_1+F\phi_2-F\phi_3=0,$$

it is evident that these equations remain true, if we put for

$$E\phi,\ E\phi+k.F\phi,$$

or for

$$H\phi,\ H\phi+k.F\phi,$$

k being any constant whatever.

If we make $k=-\sin^2\theta$, $U\phi=H\phi-\sin^2\theta.F\phi$ represents the arc of a hyperbola.

In fact, if $\frac{x^2}{\sin^2\theta}-\frac{y^2}{\cos^2\theta}=1$ be the equation to a hyperbola, and we make the ordinate $y=\cos^2\theta.\tan\phi$, we have the abscissa $x=\frac{\sin\theta}{\cos\phi}\sqrt{(1-\sin^2\theta\sin^2\phi)}$. From these we may obtain by differentiation,

$$\int\sqrt{(dx^2+dy^2)}=\int\frac{\cos^2\theta}{\cos^2\phi}\frac{d\phi}{\sqrt{(1-\sin^2\theta.\sin^2\phi)}}$$

$$=\int\frac{1-\sin^2\theta\sin^2\phi}{\cos^2\phi\sqrt{(1-\sin^2\theta.\sin^2\phi)}}.d\phi-\int\frac{\sin^2\theta.d\phi}{\sqrt{(1-\sin^2\theta.\sin^2\phi)}},$$

or $U\phi=H\phi-\sin^2\theta.F\phi$.

If we make $\sin\tau=\sin\theta.\sin\phi$, τ is the angle which the normal of the hyperbola makes with the axis of x. If we change the variable from ϕ to τ, we have

$$U=\sin^2\theta.\cos^2\theta.\int\frac{d\tau}{(\sin^2\theta-\sin^2\tau)^{\frac{3}{2}}},$$

an equation which bears a remarkable analogy to the arc of the ellipse referred to its tangent,

$$E_1-E=\cos^2\theta.\int\frac{d\tau}{(1-\sin^2\theta.\sin^2\tau)^{\frac{3}{2}}}.$$

It may be worth while to remark, that θ, the angle of the modulus, represents, in the ellipse, the eccentricity, while in the hyperbola it represents the angle between the asymptote and the ordinate.

For the comparison of hyperbolic arcs, therefore, we have the equation

$$U\phi_1+U\phi_2-U\phi_3=-\cos^2\theta.\tan\phi_1.\tan\phi_2.\tan\phi_3, \quad (9.)$$

answering to the equation for elliptic arcs,

$$E\phi_1+E\phi_2-E\phi_3=\sin^2\theta.\sin\phi_1.\sin\phi_2.\sin\phi_3. \quad (7.)$$

Formula (8.) may be derived from the equations (4.), (5.), (6.) in exactly the same way that formula (7.) is derived from the equations (1.), (2.), (3.)*.

* For the details, see LEGENDRE, 'Fonctions Elliptiques,' vol. i. p. 43, or MOSELEY "On Definite Integrals," Encyclopædia Metropolitana, 'Pure Mathematics,' vol. ii. p. 497.

In particular, if $\phi_3=\frac{1}{2}\pi$, we have, for the complementary functions, $U\phi_1+U\phi_2=\infty$, as it ought to be, since the whole length of the curve is infinite.

For duplication, making $\phi_1=\phi_2=\omega$, we have

$$U\phi_3-2U\omega=(\cos\theta.\tan\omega)^2\tan\phi_3.$$

The same formula serves for bisection, if we obtain ω from ϕ_3 by the help of the elliptic equations.

Equation (8.) is easily verified at the extremes, making $\theta=0$,

$$H\phi=\int\frac{d\phi}{\cos^2\phi}=\tan\phi,$$

and we have the known theorem

$$\tan\alpha+\tan\beta-\tan(\alpha+\beta)=-\tan\alpha.\tan\beta.\tan(\alpha+\beta).$$

If we make

$$\theta=\frac{\pi}{2},\ \ \Delta\phi=\cos\phi,\ \text{ and } H\phi=\int\frac{d\phi}{\cos\phi},$$

whence

$$\int\frac{d\phi_1}{\cos\phi_1}+\int\frac{d\phi_2}{\cos\phi_2}-\int\frac{d\phi_3}{\cos\phi_3}=0, \quad \ldots \quad (10.)$$

which is also a particular case of $F\phi_1+F\phi_2-F\phi_3=0$, depending on the particular equation

$$\sec\phi_3=\sec\phi_1\sec\phi_2+\tan\phi_1\tan\phi_2, \quad \ldots \quad (11.)$$

which I should call the MERIDIONAL EQUATION, from its connexion with the common formula for meridional parts, and with certain curves on MERCATOR'S Chart, which I have discussed elsewhere.

I have taken the trouble of deducing (8.) from (4.), (5.), (6.) directly, but the process is so exactly parallel to Mr. MOSELEY'S work, at vol. ii. p. 497 of the work above cited, that it would be unnecessary to insert it here.

A simpler verification may be found as follows: differentiating with regard to ϕ the expression $\tan\phi.\Delta\phi$, we have

$$\frac{d}{d\phi}(\tan\phi.\Delta\phi)=\frac{\Delta\phi}{\cos^2\phi}-\frac{\sin^2\theta\sin^2\phi}{\Delta\phi}=\frac{\Delta\phi}{\cos^2\phi}+\Delta\phi-\frac{1}{\Delta\phi},$$

whence, by integration (no constant needed, since each term vanishes with ϕ),

$$\tan\phi.\Delta\phi=H\phi+E\phi-F\phi. \quad \ldots \quad (12.)$$

If we now add the equations (7.) and (8.) and subtract the equation

$$F\phi_1+F\phi_2-F\phi_3=0,$$

we should have, substituting (12.),

$$\tan\phi_1.\Delta\phi_1+\tan\phi_2.\Delta\phi_2-\tan\phi_3.\Delta\phi_3=\sin^2\theta.\sin\phi_1.\sin\phi_2.\sin\phi_3-\cos^2\theta.\tan\phi_1.\tan\phi_2.\tan\phi_3. \quad (13.)$$

This equation may be easily verified by using the values of $\Delta\phi_1$, $\Delta\phi_2$, and $\Delta\phi_3$ obtained directly from the equations (1.), (2.), (3.), and clearing by means of the quadratic to which they all lead,

$$1+2\cos\phi_1.\cos\phi_2.\cos\phi_3=\cos^2\phi_1+\cos^2\phi_2+\cos^2\phi_3+\sin^2\theta.\sin^2\phi_1.\sin^2\phi_2.\sin^2\phi_3 \quad . \ . \quad (14.)$$

Equation (8.) leads to a formula for the direct reduction of the logarithmic integral of the third kind, whose parameter is negative and greater than unity. It is the exact analogue of LEGENDRE'S formula for the reduction of the same integral where the parameter is negative and less than unity, pp. 153, 154 of his third volume on Elliptic Functions. The reduction is of some importance, because on it depends the possibility of tabulating those functions, which would otherwise require a table of *treble* entry, too cumbrous to attempt.

Let ω_1 and ω_2 be two amplitudes, such that, for the common modulus θ, we have

$$\left.\begin{aligned} F\omega_1 &= F\phi + F\alpha \quad \ldots \\ F\omega_2 &= F\phi + F\alpha \quad \ldots \end{aligned}\right\} (a).$$

We must have simultaneously

$$\left.\begin{aligned} H\omega_1 + H\alpha - H\phi &= -\cos^2\theta \tan\alpha \tan\phi \tan\omega_1 \quad \ldots \\ H\phi + H\alpha - H\omega_2 &= -\cos^2\theta \tan\alpha \tan\phi \tan\omega_2 \quad \ldots \end{aligned}\right\} (b),$$

and also, putting for shortness $\delta\phi$ for $\sqrt{(1+\cos^2\theta\tan^2\phi)}$,

$$\left.\begin{aligned} \tan\omega_1 &= \frac{\tan\phi\cos\alpha\delta\alpha - \tan\alpha\cos\phi\delta\alpha}{1-\cos^2\theta\tan^2\alpha\tan^2\phi} \quad \ldots \\ \tan\omega_2 &= \frac{\tan\phi\cos\alpha\delta\alpha + \tan\alpha\cos\phi\delta\alpha}{1+\cos^2\theta\tan^2\alpha\tan^2\phi} \quad \ldots \end{aligned}\right\} (c).$$

Let us next consider the function

$$\Omega = V\omega_2 - V\omega_1 = \int\frac{d\omega_2}{\Delta\omega_2}H\omega_2 - \int\frac{d\omega_1}{\Delta\omega_1}H\omega_1.$$

If we regard α as constant, we obtain from equations (*a.*),

$$\frac{d\omega_2}{\Delta\omega_2} = \frac{d\phi}{\Delta\phi} = \frac{d\omega_1}{\Delta\omega_1},$$

whence

$$\Omega = \int\frac{d\phi}{\Delta\phi}(H\omega_2 - H\omega_1).$$

Now formulæ (*b.*) and (*c.*) give

$$H\omega_2 - H\omega_1 = 2H\alpha + \cos^2\theta\tan\alpha\tan\phi(\tan\omega_2 + \tan\omega_1),$$

and

$$\tan\omega_2 + \tan\omega_1 = \frac{2\tan\phi\cos\alpha\Delta\alpha}{1-\cos^2\theta\,.\,\tan^2\alpha\,.\,\tan^2\phi};$$

whence

$$\tfrac{1}{2}\Omega = \int\frac{d\phi}{\Delta\phi}\left\{H\alpha + \frac{\cos^2\theta\sin\alpha\tan^2\phi\delta\alpha}{1-\cos^2\theta\tan^2\alpha\tan^2\phi}\right\},$$

and, after a few reductions, we find

$$\tfrac{1}{2}\Omega = \left(H\alpha - \frac{\cos^2\theta\tan\alpha}{\Delta\alpha}\right)F\phi + \frac{\cos^2\theta\tan\alpha}{\Delta\alpha}\int\frac{1}{1-(1+\cos^2\theta\tan^2\alpha)\sin^2\phi}\cdot\frac{d\phi}{\Delta\phi};$$

or, transposing,

$$\int\frac{1}{1-(1+\cos^2\theta\tan^2\alpha)\sin^2\phi}\cdot\frac{d\phi}{\Delta\phi} - F\phi = \frac{\Delta\alpha}{\cos^2\theta\tan\alpha}\{\tfrac{1}{2}(V\omega_2 - V\omega_1) - H\alpha.F\phi\}.$$

No constant is needed, since $V\phi$ is an even function of ϕ.

One lesson we may learn from this process is, that the proper expression for the negative parameter greater than unity is $-(1+\cos^2\theta\tan^2\alpha)$. In geometrical researches this remark will probably lead to simplicity. LEGENDRE has deliberately avoided the discussion of this form of the parameter*. His reason was, that the complete integral presents itself in the form of $\infty-\infty$.

The tabulation of the function $V\omega$ would only require a table of double entry.

It may be as well to notice that the equations (*a.*), (*b.*), (*c.*) are solved by auxiliary arcs as follows:

Assume

$$\tan\eta_2=\tan\varphi\Delta\alpha,\quad \tan\eta_1=\tan\alpha\Delta\varphi,$$

then

$$\omega_1=\eta_2-\eta_1\quad,\quad \omega_2=\eta_2+\eta_1.$$

It is needless to remark that JACOBI's transformation does not enable us to reduce the integral of the circular form. The difficulty which we here encounter, is exactly analogous to that which presents itself in the reduction of the cubic equation of ordinary algebra. In fact, if we were to apply JACOBI's transformation to *one only* of α or φ, the auxiliary arcs just mentioned would give values of ω of the form $\eta\pm\eta'\sqrt{-1}$, and the difficulty would depend upon the interpretation of $F(\eta\pm\eta'\sqrt{-1})$.

* See Fonctions Elliptiques, vol. i. p. 71. sect. 53.